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Large-Scale Structure of Isotropic Homogeneous Turbulence

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It is shown that the longitudinal correlation function f is asymptotically proportional to r^{-3} as $r \to \infty$ and the energy spectrum function is asymptotically proportional to κ^2 as $\kappa \to 0$ if and only if $0 < \langle (\int \mathbf{u}\, d^3x) \cdot \mathbf{u} \rangle < \infty$. Moreover, the latter finiteness condition is shown to be essentially equivalent to $\langle (\int \mathbf{y} \cdot \mathbf{u}\, d^3x)^2 \rangle < \infty$ for nonstochastic $\mathbf{y} \in L^2(R_3)$. Confirmed by recent experimental measurements, the large r dependence $f \propto r^{-3}$ is concomitant with an $O(r^{-6}) = O(f^2)$ fall-off of the viscous force term in the Kármán–Howarth equation.

1. BASIC RELATIONS

Consider isotropic homogeneous incompressible fluid turbulence in a Galilean frame such that the mean velocity vanishes. Isotropy, homogeneity, and incompressibility imply that the two-point correlation tensor has the generic forms (Batchelor, 1960)

$$
\begin{aligned}
R_{jk}(\mathbf{r}, t) &\equiv \big\langle u_j(\mathbf{x}', t) u_k(\mathbf{x}'', t) \big\rangle \\
&= u^2 \left[\left(f + \frac{r}{2} \frac{\partial f}{\partial r} \right) \delta_{jk} - \frac{1}{2r} \frac{\partial f}{\partial r} r_j r_k \right] \\
&= \int \frac{E(\kappa, t)}{4\pi\kappa^2} \left(\delta_{jk} - \frac{\kappa_j \kappa_k}{\kappa^2} \right) e^{i\boldsymbol{\kappa} \cdot \mathbf{r}} d^3\kappa \qquad (1)
\end{aligned}
$$

where $u^2 = u^2(t)$, $\mathbf{r} \equiv \mathbf{x}' - \mathbf{x}''$, $r \equiv |\mathbf{r}|$, and the longitudinal correlation function $f = f(r, t)$ is normalized to give $f(0, t) \equiv 1$. The energy spectrum function $E(\kappa, t)$ is nonnegative for all wavenumbers $\kappa \equiv |\boldsymbol{\kappa}|$ in the final member of (1).

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Suppose that the global expectation value

$$\left\langle \left(\int \mathbf{u}(\mathbf{x}',t)\,d^3x' \right) \cdot \mathbf{u}(\mathbf{x}'',t) \right\rangle \equiv \mathfrak{M}(t) \tag{2}$$

exists as a finite positive quantity. Then it follows immediately from (1) that

$$\mathfrak{M}(t) = \int R_{jj}(\mathbf{r},t)\,d^3r = u^2 \int_0^\infty \left(3f + r\frac{\partial f}{\partial r} \right) 4\pi r^2\,dr$$

$$= 4\pi u^2 \lim_{r\to\infty} (r^3 f) = 4\pi^2 [E(\kappa,t)/\kappa^2]_{\kappa=0} \tag{3}$$

and hence one obtains (Rosen, 1981)

$$f(r,t) \doteq I(t) r^{-3} \qquad \text{as } r \to \infty \tag{4}$$

$$E(\kappa,t) \doteq (4\pi^2)^{-1} \mathfrak{M}(t) \kappa^2 \qquad \text{as } \kappa \to 0 \tag{5}$$

where $I(t) \equiv (4\pi u^2)^{-1} \mathfrak{M}(t)$.

The complementary asymptotic forms (4) and (5) are in agreement with experimental measurements and have also appeared in previous theory. Energy spectra with $E(\kappa,t) \propto \kappa^2$ for small κ were found experimentally by Stewart and Townsend (1951) [see their Figure (10)]. The latter experimental evidence was discussed in a general theoretical context by Birkhoff (1954). For statistically steady turbulent fluid motion maintained by statistically steady random impulsive forces, Saffman (1967) obtained (4), (5) with $I(t) \equiv$ const. Most recently, the measurements of Frenkiel et al. (1979) support the asymptotic dependence $f(r,t) \propto r^{-3}$ for large r (see Appendix).

Hence, the postulate that (2) exists as a finite positive quantity implies a large-scale turbulence structure consistent with experiments. As a statistical property or postulate, however, the finiteness of (2) is complicated because the left side is asymmetric in $\mathbf{u}$ and the dot product is indefinite in sign over the ensemble. The principal purpose of the present communication is to show that the finiteness of (2) is essentially equivalent to the finiteness of the global expectation value displayed in (10) below. Stated precisely, (10) guarantees validity of the asymptotic forms (4), (5) with $0 \leq \mathfrak{M}(t) < \infty$. Compared to (2), the global expectation value in (10) is more desirable conceptually, for it is the positive-definite square of a real scalar integral.

2. EQUIVALENCE THEOREM

Associated with the two-point correlation tensor (1) is the eigenvalue equation

$$\int R_{jk}(\mathbf{x}'-\mathbf{x}'', t)\zeta_k(\mathbf{x}'')\, d^3\mathbf{x}'' = \lambda\zeta_j(\mathbf{x}') \tag{6}$$

in which $\boldsymbol{\zeta}(\mathbf{x})$ is a complex-valued vector field and $\lambda = \lambda(t)$ is a real scalar function of t that depends on R_{jk} and $\boldsymbol{\zeta}$. By employing the final member of (1) and making use of the convolution theorem for Fourier integrals, (6) is readily solved to yield the eigenfunctions and eigenvalues

$$\boldsymbol{\zeta}(\mathbf{x}) = \mathbf{a}e^{i\mathbf{b}\cdot\mathbf{x}} \qquad \lambda = 2\pi^2[E(\kappa, t)/\kappa^2]|_{\kappa=|\mathbf{b}|} \tag{7}$$

in which the real constant parameter vectors $\mathbf{a}$, $\mathbf{b}$ are perpendicular ($\mathbf{a}\cdot\mathbf{b}=0$) but otherwise arbitrary. In addition to (7), the eigenvalue equation (6) also admits longitudinal vector eigenfunctions

$$\boldsymbol{\zeta}(\mathbf{x}) = \mathbf{b}e^{i\mathbf{b}\cdot\mathbf{x}} \qquad \text{with } \lambda = 0 \tag{8}$$

Since all solutions to (6) are either of the form (7) or (8), the eigenvalue spectrum of $R_{jk}(\mathbf{r}, t)$ is compact [i.e., $0 \leqslant \lambda \leqslant \lambda_{\max}(t)$] if $\lambda_{\max}(t) = 2\pi^2 \max_\kappa (E(\kappa, t)/\kappa^2)$ is finite. Because $E(\kappa, t)$ is finite or zero for all κ, $\lambda_{\max}$ is likewise finite if and only if $[E(\kappa, t)/\kappa^2]_{\kappa=0} \equiv (4\pi^2)^{-1}\mathfrak{M}(t)$ is finite or zero. It then follows from elementary Hilbert-space theory (e.g., Akhiezer and Glazman, 1961) that the real symmetric tensor kernel $R_{jk}(\mathbf{r}, t)$ is bounded,

$$\int R_{jk}(\mathbf{x}'-\mathbf{x}'', t)y_j(\mathbf{x}')y_k(\mathbf{x}'')\, d^3x'\, d^3x'' \leqslant \lambda_{\max}(t)\int |\mathbf{y}(\mathbf{x})|^2\, d^3x \tag{9}$$

for any real $\mathbf{y}(\mathbf{x}) \in L^2(R_3)$. Conversely, if

$$\sup_{\mathbf{y}}\left[\left\langle\left(\int \mathbf{u}(\mathbf{x}, t)\cdot\mathbf{y}(\mathbf{x})\, d^3x\right)^2\right\rangle\Big/\int |y(\mathbf{x})|^2\, d^3x\right] \equiv \lambda_{\max}(t) \tag{10}$$

is finite, then $\mathfrak{M}(t) = 4\pi^2[E(\kappa, t)/\kappa^2]_{\kappa=0}$ must be finite or zero. Although the $\mathfrak{M}(t)=0$ special case is not precluded by the finiteness of (10), the latter condition is otherwise equivalent to (2). Thus the finiteness of (10) implies the asymptotic forms (4), (5) with $0 \leqslant \mathfrak{M}(t) < \infty$.

3. DYNAMICAL INTERPRETATION

For freely decaying Navier–Stokes isotropic homogeneous turbulence, one has the Kármán–Howarth (Kámán and Howarth, 1938) equation

$$\frac{\partial}{\partial t}(u^2 f) = 2\nu u^2 \left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r} \right) f + K \tag{11}$$

in which ν is the constant kinematic viscosity and $K = K(r, t)$ is defined implicitly by the triple-velocity correlation

$$\left\langle u_k(\mathbf{x}+\mathbf{r}, t) u_j(\mathbf{x}, t) u_k(\mathbf{x}, t) \right\rangle = \tfrac{1}{2} r_j K \tag{12}$$

Suppose that f is analytic in r^{-1} about $r = \infty$ and has the asymptotic dependence shown in (4) [as exhibited, for example, by (A1) below]. Then

$$f(r, t) = I(t) r^{-3} + O(r^{-4}) \tag{13}$$

and (11) becomes

$$\frac{\partial}{\partial t}(u^2 f) = O(r^{-6}) + K \tag{14}$$

by substituting (13) into the viscous force term. Conversely, the asymptotic dependence (13) follows from the requirement that f be analytic in r^{-1} and that the viscous force term in (11) be of $O(r^{-6})$. Compatible with the finiteness of (2) and (10), the $O(r^{-6}) = O(f^2)$ fall-off of the viscous term in (11) is a dynamical effect related to the small-scale structure and the existence of the Taylor microscale length of order $(\nu t)^{1/2}$.

Finally it should be noted that (13), (14), and the definition below (5) yield

$$K = (4\pi)^{-1} [d\mathcal{M}(t)/dt] r^{-3} + O(r^{-4}) \tag{15}$$

The general dynamics of decay (e.g., Rosen, 1980) indicates that $\mathcal{M}(t)$ decreases monotonically with increasing t, in the same manner as (but not necessarily proportional to) $u^2 = u^2(t)$. Hence, the leading term in (15) does not vanish in general, and K is therefore of $O(r^{-3}) = O(f)$ for freely decaying Navier–Stokes turbulence.

TABLE I. Comparison of Experimental Values for the Longitudinal Correlation Function Obtained by Frenkiel et al. (1979)
[Their Figure 2 with the Taylor (1938) approximation $f \cong R(r/U)$]
and the Values Given by the Empirical Relation (A1)

r/M	0	0.10	0.20	0.30	0.40	0.60	1.00	1.60	2.00	2.40	2.80	3.20
$f[\cong R(r/U)]$	1	0.80	0.65	0.52	0.45	0.32	0.19	0.09	0.06	0.04	0.03	0.02–0.03
f[by (A1)]	1	0.800	0.651	0.536	0.447	0.320	0.180	0.090	0.061	0.043	0.032	0.024

APPENDIX. EMPIRICAL FORM OF THE LONGITUDINAL CORRELATION FUNCTION FOR LARGE REYNOLDS NUMBER GRID-GENERATED TURBULENCE

For grid-generated turbulence at Reynolds number UM/ν from 12,800 to 81,000 and typical turbulence levels $u/U \sim .02$ in air and water, Frenkiel et al. (1979) have observed the longitudinal correlation function to be independent of t and to depend exclusively on the dimensionless geometrical ratio (r/M). This beautiful universality is expressed by the empirical relation (Rosen, 1981)

$$f(r,t) = [1 + 0.770(r/M)]^{-3} \tag{A1}$$

as shown by the comparison in Table I. Clearly (A1) is consonant with the asymptotic dependence in (4) for large r. Moreover, it follows from (A1) that the prefactor in (4) $I(t) = 2.19M^3$ is identically constant, as in the theoretical model of Saffman (1967).

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